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A CHARACTERIZATION OF NORMAL OPERATORS†

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ABSTRACT
Let A be a bounded linear operator in a Hilbert space. If A is normal then $\log\|e^{At}u\|$ and $\log\|e^{A^*t}u\|$ are convex functions for all $u\neq 0$. In this paper we prove that these properties characterize normal operators.

1. Introduction

Let H be a Hilbert space over the complex numbers C with an inner product (x,y). Assume that $A : H \to H$ is a bounded linear operator. A straightforward calculation shows (see the next section)

LEMMA 1. *Let $A : H \to H$ be a bounded linear operator. If $A^*A - AA^*$ is non-negative definite then $\log\|e^{At}u\|$ is convex on R for all $u\neq 0$.*

Thus if A is normal then $\log\|e^{At}u\|$ and $\log\|e^{A^*t}u\|$ are convex. However, there are non-normal operators A such that $0 \leqq A^*A - AA^*$. Here, as usual, for self-adjoint operators S, T the inequality $S \leqq T$ denotes that $T - S$ is a non-negative definite operator. For example let $H = l_2$ and choose A to be the shift operator $A(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots)$. In this case $\log\|e^{A^*t}u\|$ is not convex for $u = (0,1,0,\cdots)$. This situation can not hold in a finite dimensional H. More precisely we have

THEOREM 1. *Let $A = P + iQ$, where P and Q are bounded self-adjoint operators. Assume that P has only a point spectrum (i.e. H has an orthonormal basis consisting of eigen-elements of P). Then A is normal if and only if*

$$(1) \qquad \frac{d^2}{dt^2}(\log\|e^{At}u\|)(0) \geqq 0, \qquad \textit{for all } u \neq 0.$$

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Our main result is

THEOREM 2. *Let $A : H \to H$ be a bounded linear operator. Then A is normal if and only if* (1) *and*

$$\frac{d^2}{dt^2}(\log\|e^{A^*t}u\|)(0) \geqq 0, \qquad \text{for all } u \neq 0, \tag{2}$$

hold.

We conjecture

CONJECTURE. *Assume that* (1) *holds. Then* $0 \leqq A^*A - AA^*$.

2. Proofs

Using the group properties of e^{At} we easily deduce

LEMMA 2. *Let $A : H \to H$ be a bounded linear operator. Then* $\log\|e^{At}u\|$ *is convex on R for all $u \neq 0$ if and only if* (1) *holds.*

A straightforward calculation shows

$$\frac{d^2}{dt^2}(\log\|e^{At}u\|)(0)$$
$$= \tfrac{1}{2}(u,u)^{-2}[((A^2 + A^{*2} + 2A^*A)u,u)(u,u) - ((A + A^*)u,u)^2].$$

Thus (1) is equivalent to the inequality

$$((A + A^*)u,u)^2 \leqq ((A^2 + A^{*2} + 2A^*A)u,u)(u,u). \tag{3}$$

The Cauchy–Schwarz inequality yields

$$((A + A^*)u,u)^2 \leqq ((A + A^*)^2u,u)(u,u).$$

As

$$(A + A^*)^2 = A^2 + A^{*2} + 2A^*A - (A^*A - AA^*)$$

the assumption that $A^*A - AA^* \gneqq 0$ implies the inequality (3). This establishes Lemma 1.

To give an equivalent form of the inequality (3) we need the following lemma.

LEMMA 3. *Let $R, S, T : H \to H$ be self-adjoint non-negative definite operators. Then*

$$(Ru,u)^2 \leqq (Su,u)(Tu,u), \qquad \text{for all } u \in H \tag{4}$$

if and only if

$$2R \leqq \alpha^{-1}S + \alpha T \tag{5}$$

for all positive α.

PROOF. The inequality (4) implies (5) in view of arithmetic-geometric inequality. Suppose that (5) holds. If $(Su, u) = 0$ then by letting α tend to zero we deduce that $(Ru, u) = 0$. Thus we may assume that $(Su, u)(Tu, u) > 0$. In that case choose $\alpha = [(Su, u)/(Tu, u)]^{\frac{1}{2}}$ to obtain (4). ■

LEMMA 4. *Let* $A = P + iQ$, *where* P *and* Q *are self-adjoint. Then* (3) *is equivalent to the inequality*

$$\frac{i}{2}(QP - PQ) \leqq (P - \alpha I)^2 \tag{6}$$

for all real α.

PROOF. A straightforward computation shows that the inequality (3) is invariant under the transformation $A \to A + \omega I$. So we may assume that $P \geqq 0$. Also, in terms of P and Q, (3) becomes

$$(Pu, u)^2 \leqq \left(\left[P^2 + \frac{i}{2}(PQ - QP)\right] u, u\right)(u, u).$$

In view of Lemma 3 the above inequality is equivalent to (6) for $\alpha > 0$. As $P \geqq 0$, (6) trivially holds also for $\alpha \leqq 0$. Again (6) is invariant under the transformation $A \to A + \omega I$. The proof of the lemma is completed. ■

LEMMA 5. *Let* $P, Q : H \to H$ *be bounded self-adjoint operators. Assume that* $Pu = \alpha u$, $u \neq 0$ *and suppose that* (6) *holds. Then*

$$P(Qu) = \alpha(Qu). \tag{7}$$

PROOF. Let $y = u + sx$, where $s \in C$ and $(u, x) = 0$. As

$$(Bu, u) = ((P - \alpha I)^2 u, u) = ((P - \alpha I)^2 u, x) = 0, \qquad B = \frac{i}{2}(QP - PQ),$$

(6) implies

$$2\,\mathrm{Re}\{\bar{s}(Bu, x)\} + |s|^2(Bx, x) \leqq |s|^2((P - \alpha I)^2 x, x).$$

Since s is arbitrary we obtain that $(Bu, x) = 0$ if $(u, x) = 0$. So $Bu = \beta u$. Finally the equality $(Bu, u) = 0$ yields $\beta = 0$, i.e., $Bu = 0$. This proves (7). ■

PROOF OF THEOREM 1. As P has only a point spectrum H decomposes to a direct sum of invariant eigen-subspaces of P:

$$H=\sum_{\lambda\in\sigma(P)}\bigoplus H_\lambda,\qquad (P-\lambda I)H_\lambda=0.$$

Lemma 5 implies that $QH_\lambda\subset H_\lambda$. That is $PQ=QP$ which is equivalent to the normality of A. ∎

Assume now that $\log\|e^{A t}u\|$ and $\log\|e^{A^* t}u\|$ are convex on R for all $u\neq 0$. According to Lemma 4 these conditions are equivalent to

$$-(P-\alpha I)^2\leqq\frac{i}{2}(QP-PQ)\leqq(P-\alpha I)^2 \tag{8}$$

for all $\alpha\in R$. Then Theorem 2 follows from our last theorem.

THEOREM 3. *Let B, $P:H\to H$ be bounded self-adjoint operators. Assume that*

$$-(P-\alpha I)^\mu\leqq B\leqq(P-\alpha I)^\mu,\qquad \mu=2m/(2l-1) \tag{9}$$

for all real α, where $m\geqq l\geqq 1$ are integers. Then $B=0$.

PROOF. Suppose that $Pu=\alpha u$. Then (9) yields $(Bu,u)=0$. Apply the arguments of the proof of Lemma 5 to deduce $Bu=0$. Decompose $H=H_1\bigoplus H_2$, $PH_i\subseteq H_i$ such that H_2 has an orthonormal basis consisting of eigen-elements of P and H_1 — the orthogonal complement of H_2 — does not contain any eigen-elements of P. Thus $BH_2=0$. Therefore it is enough to assume that P has only a continuous spectrum. Without restriction in generality we may assume that the spectrum of P lies in $[0,1]$. Consider the spectral decomposition of P

$$P=\int_0^1\lambda dE(\lambda).$$

Let

$$E_i=\int_{(i-1)/n}^{i/n}dE(\lambda),\qquad i=1,\cdots,n.$$

Thus

$$I=\sum_{i=1}^n E_i,\quad E_iE_j=\delta_{ij}E_j,\qquad i,j=1,\cdots,n.$$

Choose $\alpha=(2i-1)/2n$. Then (9) yields

$$-(2n)^{-\mu}E_i\leqq E_iBE_i\leqq(2n)^{-\mu}E_i. \tag{10}$$

Let $y=u+sy$, $u\in E_iH$, $y\in(I-E_i)H$. Then for the same choice of α, (9) implies

$$|(Bu,u)+2\operatorname{Re}\{s(By,u)\}+|s|^2(By,y)|\leqq(2n)^{-\mu}(u,u)+|s|^2(y,y).$$

The same inequality applies if we replace s by $-s$. Combine these two inequalities to get

$$2|\mathrm{Re}\{s(By,u)\}| \leqq (2n)^{-\mu}(u,u)+|s|^2(y,y).$$

Choose $|s|=(2n)^{-\mu/2}$, $\arg s = -\arg(By,u)$ to deduce

(11) $\quad |(By,u)| \leqq (2n)^{-\mu/2}[(u,u)+(y,y)]/2, \qquad u \in E_iH, \quad y \in (I-E_i)H.$

Let $\lambda \in \sigma(B)$. We claim that

(12) $$|\lambda| \leqq 3(2n)^{-(\mu-1)/2}.$$

Indeed, there exists $x \in H$ such that

$$\|Bx-\lambda x\| \leqq (2n)^{-\mu/2}, \qquad \|x\|=1.$$

As $\|x\|^2=\Sigma_{i=1}^n\|E_ix\|^2=1$ we may assume that $\|E_jx\| \geqq n^{-\frac{1}{2}}$ for some $1 \leqq j \leqq n$. So

$$\|E_jBx-\lambda E_jx\| \leqq (2n)^{-\mu/2}.$$

Thus

$$|\lambda| \leqq \sqrt{n}((2n)^{-\mu/2}+\|E_jBx\|).$$

We now estimate $\|E_jB\|$. Clearly

$$\|E_jB\| = \sup_{\|v\|=\|w\|=1} \mathrm{Re}\{(E_jBv,w)\} = \sup_{\|v\|=\|E_jw\|=1} \mathrm{Re}\{(E_jBv,E_jw)\}$$

$$\leqq \sup_{\|E_jv\|=\|E_jw\|=1} \mathrm{Re}\{(E_jBE_jv,E_jw)\} + \sup_{\|(I-E_j)v\|=\|E_jw\|=1} \mathrm{Re}\{(E_jB(I-E_j)v,E_jw)\}.$$

In view of (10) and (11) we get

$$\sup_{\|E_jv\|=\|E_jw\|=1} \mathrm{Re}\{(E_jBE_jv,E_jw)\} \leqq (2n)^{-\mu},$$

$$\sup_{\|(I-E_j)v\|=\|E_jw\|=1} \mathrm{Re}\{(E_jB(I-E_j)v,E_jw)\} \leqq (2n)^{-\mu/2}.$$

Combine the above inequalities to deduce (12). As n is arbitrary and $\mu>1$, (12) implies $\sigma(B)=\{0\}$.

As B is self-adjoint we conclude that $B=0$.

Added in proof. It was pointed out by C. Foiaş that, following the results of the paper, an operator A from a Banach space B to itself is defined to be normal if $\log\|e^{At}u\|$ and $\log\|e^{A^*t}g\|^*$ are convex for all $0 \neq u \in B$, $0 \neq g \in B^*$($\|\cdot\|^*$ is the conjugate norm on B^*). Furthermore a normal A is called hermitian if the spectrum of A is contained in the real line. The properties of these operators will be studied elsewhere.

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